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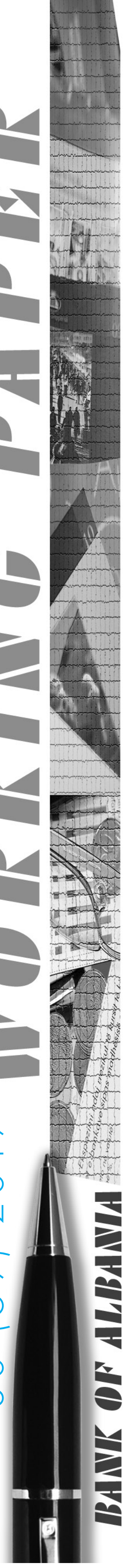

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Note: The authors would like to emphasize that the ideas and comment expressed in this paper are responsibility of the authors only and not those of the Bank of Albania.

C O N T E N T

ABSTRACT

This paper continues the development of the multi-dimensional density as alternative method of empiric investigation in the study of economics as a social phenomenon, defined and considered as a random event. The paper further develops the research of Tanku and Ceca (2013) by providing the tools and the metric of estimation method. The paper is based on the definition of the economy as a multi-dimensional random process and the estimation of the multi-dimensional joint probability function. The empirical relevance of the methodology is demonstrated by the reexamination of the relationship between money and inflation in the case of developed economy. This alternative is very important because it is free of several restricting assumptions about characteristics of the data end errors and ex-ante knowledge of theoretic relationship and the functional form, and the traditional hypothesis testing framework of empiric research.

1. INTRODUCTION

Economics has made substantial advance in the development of empiric methods for the investigation of economic phenomenon. The use of mathematic models has become the mainstream and the fundamental tool in of theoretic and empiric research for understanding and estimating economic relationships. This has led to the development of several methods of estimation for theoretically founded relationships among economic variables. Most of the work currently in empiric research is based on time series analysis and similar techniques which like theoretic models require at least some sort of ex-ante knowledge and ex-ante assumptions regarding the model structure and model specification. In fact all empiric models are based to some extend on the ex-ante assumptions and hypothesis about economic theory and theoretic relationships between economic variables. Therefore the objective of empiric investigation is to test these hypothesis regarding model selection and model specification, rather than suggesting potential direction and form of economic relationships. In addition current methods require certain pre testing and data transformation, a set of rigid assumptions about the property of the dataset and of the estimated errors. The main characteristic of all these mainstream methods of estimation is that they consider economic variables as vectors of data which elements are realizations of the variables that corresponds to each time (day, week, month, quarter or year) during the entire period of investigation. The methods test the relationships in terms of correlation, cointegration, etc. among these vectors (of variables).

In their paper Tanku & Ceca (2013) propose and introduce a new alternative approach for empiric research in economics. This paper proposes the tools and the metric for interpretation of economic relationships among economic variables of interest. This is illustrated in the empirical investigation of the money inflation relationship. The focus of the paper is on the methodology rather than on the economic relationship per se. The empirical relationship is included in the paper as means of illustration the methodology rather than test economic theory though it makes a good job in explaining the theory.

The method (Multi-Dimensional Density Estimation) proposed by Tanku and Ceca (2013) portrays and visualizes the economy as random event. It builds upon the definition of random events and data generating process (DGP) of Ericsson, Hendry and Mizon (1998), who portray (formalize the definition of) the economy (or economic behavior) in the form of a multiple dimensional random process. This DGP is defined in the form of a probability space *[Ω, F, P(.)]*, where *Ω* represents a sample space of the vectors of d variables, *F* is the event space, and *P(·)* is the probability measure for the events in *F*. Following this definition we use the density estimation as an alternative method of investigation. This is achieved by the definition of social events like economics as multidimensional simultaneous random events that result from the interaction of many economic variables each embodied with a certain probability density function.

Tanku and Ceca (2013) write:

> *… following the DGP that results from the random event we call economy, one can reasonably define the economy as a d dimensional space generated by d different random variables in which each m (m < d) dimensional space represents a subspace of the entire space of our random event. If we were to scale all these possible* ξ_m *spaces in their relationship we will get the follows:* $\xi_1 \subseteq \xi_2 \subseteq \cdots \subseteq \xi_m \subseteq \cdots \subseteq \xi_d$*, which in substance represents an expanding sequence of spaces of the economy. It would mean that starting from each following event would represent a new possible expansion space of the original event. Now assuming that all added variables are linearly depended (or endogenous) to first chosen variables, it means that the new space will carry itself and so preserve the same DGP along fewer dimensions. If one or more of the new added variables independent from the rest (or exogenous), than the distribution of the new space will also 'carry' this attribute to its distribution. These are all the conditions that we have to judge through joint densities of the variables create those spaces. (pp. 11)*

This allows any empirical study to extend this "economy" to as many dimensions as there are variables in the economy (sample) representing the event that results from du to the interaction among economic variables that we observe as the set of economic data. Tanku and Ceca (2013) show that this alternative approach offers many benefits compared to traditional methods of empiric investigation. The main benefit is the departure from the traditional methodology of hypothesis testing. The method has also many other advantages relative to traditional tools of empiric research due to its departure from several assumptions about normality, or transformation of the data in logs but most important the ex-ante knowledge about the functional form or model specification.

This is made possible due to the fact that different from current methods focus on F as a source of information for the matrix of coefficients that control the interrelationship among simultaneous and past values of the variable of interest, Tanku and Ceca (2013) suggests to extract the same information from the *P(.)*. This is made possible via the calculation of the multi-dimensional probability as a way measure the probability of events in *F*. The measured densities are then using this measure to interpret and analyze the economy. This analyze is based on the numerical characteristics of the estimated probability densities as the tool for interpretation of the relationship among the variables of interest.

The economy takes place as a stochastic multi-dimensional event however bounded by our three dimensional perception we could only perceive (view) it as a set of two dimensional graphs plus the third dimension upon which density is measured. For this purpose we take projections of the d dimensional space into two dimensional spaces and use the shape of the estimated distributions to interpret the relationship among corresponding variables. In other words the methods call for reading and interpreting the graphical presentation form of the Estimated JPDF. Therefore Tanku and Ceca (2013) propose the interpretation of the projections of this multidimensional event in the three dimensional space. Which means, take variables in couples and studying their behavior in the resulting three dimensional diagrams of joint density functions. Plotting two dimensional density estimations of the desired variable with the rest of the *d – 1* other variables reveals the behavior of the variable of interest relative to the other chosen explanatory variables. From here the analysis can expand to include a set of one and/or multidimensional consecutive constrains in any of the *d – 1* to see how the variable of interest changes in response to changes in the other variables of our economy. The final conclusion is drawn on the changes in the resulting constrained distributions (or their moments).

Tanku and Ceca 2013 employ the method to study the relationship of inflation and a set of economic variables in the case of Albania.

> *Generally the analysis for each variable of interest x_m is based on the comparison and the evolution of the moments of conditional distributions of x_m. The method is used on differenced data (in order to remove potential build in common time or stochastic trends as is commonly practice in time series) or percentage changes in variables with respective distributions potentially revealing information regarding structural relationships or the contemporaneous links among variables and finally growth rates respectively. Tanku and Ceca (2013) describe other potential use of the method which allows information extracted from JPDF to potentially provide the researcher with the equivalent of causality, structural analysis, and impulse response and variance decomposition which is the set of information provided by the current empiric investigation methods.*

The conclusions in their paper (Tanku & Ceca (2013)) are however drawn based on "naive interpretation" of the general shape of the joint distribution of inflation and the other variables of interest. The intuitive interpretation of the relationship is similar to interpretation of a scatter diagram but with a fundamental difference. Rather than interpreting or finding the path which has the least distance from all pints in the graph the density estimation extract the information from the probability measure of the random event and the path is not a distance but a weighted measure of probability mass, and it can be expressed in terms of numerical characteristics if the resulting distribution. The advantage of this method however is that densities provide a detailed description of the phenomenon while at the same time they come with a well-defined metric (numerical characteristics of the distributions). However our analysis suffered from the fact that we had not yet developed the set of tools and the metric which will allow researcher to read and interpret the changes in the shape of estimated densities.

In this paper we propose the intersection as a method (tool) to read (extract) information from the JPDF and the numerical characteristics of the PDF of the intersection to analyze and compare the behavior of the variables of interest in response changes in other variables of the economy.

Therefore if we were able to "read" these two dimensional densities as conditional densities (conditioned for any particular value of the independent variable) of the dependent variable we can measure and interpret its behavior based on the interpretation of the numerical characteristics (or the moments) of all (continuous) one dimensional probability distribution of the dependent variable. We call these conditional probabilities the "cross section" or "intersection". The process can repeated for all (for each and every one of the other) $d - 1$ dimensions (variables) in the economy. Finally observing the changes in the moments of the resulting conditional distributions (as we move along different values of the variables in the other dimensions) one can understand how our variable of interest evolves in response to changes in the explanatory variables.

This paper enhances the previous work by proposing a method to by introducing the tools and metric of interpretation of the multi-dimensional calculate estimate conditional probabilities and use it as a tool to read the information portrayed by the two dimensional graphs presented in our previous work. This will enhance the understanding beyond simple mapping of the relationship and the naive interpretation of these maps and allow the researcher to interpret and forecast the behavior of the variable (dimension) of interest when the values of other variables are known or exogenously forecasted.

Therefore the rest of the paper will deal with the definition of the "cross section" and its interpretations.

2. INTRODUCING THE "CROSS SECTION" METHOD AS A TOOL AND METRIC OF INVESTIGATION

The joint density probability function is an easy way to deal with traditional question in the study of economics in terms of definitions for causality and endogeneity, etc. This method investigates the economy in its graphical presentation of the d dimensions through its projection into two dimensional spaces. The resulting three dimensional graphs contain the entire information regarding the probabilistic relationship among our variables of interest. The interpretation of these results was however based on the information extracted from the resulting probability functions represented in the three dimensional graphs described above.

In this section we define, establish and provide mathematic proof and finally explain the tool of conditional probabilities (cross section) as the tool of empiric investigation. This is the cross-section of the graph of the joint density function, at each (or elected) unit(s) of the independent variable(s) resulting in a density function. In this respect cross section represents conditional probabilities of the variable of interest for any value *n* of the other *d - 1* explanatory variables, that comprise our economy. The method yields in total *n x d-1* conditional probabilities which simultaneously provide information provide information regarding the relationship among the variables of interests.

DEFINITION 1:

Cross section is defined as: Generalized definition of conditional distribution of a continuous random variable – the case when the condition is a set to be a single value (point intersection or cross section).

Let us assume that there have been n realizations of the d dimensional variable $\underline{X}$, respectively $\underline{x}_1$, $\underline{x}_2$, ..., $\underline{x}_n$ As vectors, the sample can be written:

$$\begin{cases} \underline{x}_1 = \left(x_1^1, x_1^2, \ldots, x_1^d\right) \\ \underline{x}_2 = \left(x_2^1, x_2^2, \ldots, x_2^d\right) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \underline{x}_n = \left(x_n^1, x_n^2, \ldots, x_n^d\right) \end{cases} \qquad \textit{alternatively} \quad X_{1\times n} = \mathcal{F}_{d\times n} \tag{1}$$

In that meaning,

x_i where $i \in \{1,2, \ldots, n\}$ represents the state of the random variable (in our case the economy) at time *I*; $\mathcal{F}$ represents the event space of the DGP, therefore the realizations given in each column *k* of $\mathcal{F}$ where, $k \in \{1,2, \ldots, d\}$ represent the time series vectors of the of the d variables of the economy for the entire period if investigation, and any element x_i^k of the $\mathcal{F}$ matrix represents the observation of the variable *k* at time *i*.

Let also assume that $\underline{t} = (t_1, t_2, \ldots, t_d)$ element of R^d, is the variable (argument) of the density which will be estimated.

In literature posed that the density estimation with kernel $K(\underline{t})$, is given as follows:

$$\hat{f}(\underline{t}) = \frac{1}{nh^d} \cdot \sum_{i=1}^{n} K\left(\frac{1}{h}\left(\underline{t} - \underline{x}_i\right)\right), \tag{2}$$

where: $t \in R^d$; $x_1, x_2, \ldots, x_n$ are the elements of sample from R^d; and $\hat{f}(\underline{t}) \in R$.

Now we can define point intersection of the density estimates, in other words, conditional densities of the density estimates, with point intersection in the one and **d** dimensional case

In the following we present and calculate the conditional densities of the density estimates in line with definition 1. The "conditionality" of the densities is assumed based in the standard definition of the conditional density.

Let's suppose that the condition is the variable t_k, for $k \in \{1,2, \ldots, d\}$.

Based on the standard definition of the probability theory, the expression of the conditional distribution is:

$$\hat{f}\left({}^{\underline{t}}/_{t_k}\right) = \frac{\hat{f}(\underline{t})}{\hat{f}_{\{k\}}(t_k)} \tag{3}$$

where:

$\hat{f}\left({}^{\underline{t}}/_{t_k}\right)$ – is the conditional density estimate with the condition the variable t_k.

$\hat{f}_{\{k\}}(t_k)$ – is the marginal density of the density estimate, for the variable t_k.

Following we calculate the conditional density estimates (3) for the case when the density estimates are compiled using the Normal and Epanechnikov kernels, with conditions 1 and d dimensional. We posed the results in the form of four theorems.

Theorem 1: *The conditional density estimate with Gaussian Kernel, with 1 dimensional condition is:*

$$\hat{f}\left({}^{\underline{t}}/_{t_k}\right) =$$

$$= \frac{1}{h^{d-1}(2\pi)^{\frac{d-1}{2}}} \cdot \frac{\sum_{i=1}^{n} exp\left\{-\frac{1}{2h^2}\overbrace{\left[\left(t_1 - x_i^1\right)^2 + \ldots + \left(t_k - x_i^k\right)^2 + \ldots + \left(t_d - x_i^d\right)^2\right]}^{d\ monoms\ (terms)}\right\}}{\sum_{i=1}^{n} exp\left[-\frac{1}{2h^2}\left(t_k - x_i^k\right)^2\right]}$$

Proof:

Density estimate $\hat{f}(\underline{t})$ below is given by Tanku Ceca (2013) eq. 4.2.3:

$$\hat{f}(\underline{t}) = \frac{1}{nh^d(2\pi)^{d/2}} \cdot \sum_{i=1}^{n}\left\{exp\left[-\frac{1}{2h^2}(t_1 - x_i^1)^2\right] \cdot \ldots \cdot exp\left[-\frac{1}{2h^2}\left(t_k - x_i^k\right)^2\right] \cdot \ldots \cdot exp\left[-\frac{1}{2h^2}\left(t_d - x_i^d\right)^2\right]\right\}$$

Further let's calculate the denominator at (3), means marginal density of the density estimate, which is function of the variable t_k:

$$\hat{f}_{\{k\}}(t_k) = \int_{R^{d-1}} \hat{f}(\underline{t})\, dt_1 \dots dt_{k-1}\, dt_{k+1} \dots dt_d == \frac{1}{nh^d (2\pi)^{d/2}} \sum_{i=1}^{n} \Bigg(exp\left[-\frac{1}{2h^2}\left(t_k - x_i^k\right)^2\right] \cdot$$
$$\Bigg\{ \underbrace{\int_R exp\left[-\frac{1}{2h^2}(t_1 - x_i^1)^2\right] dt_1}_{=I} \cdot \dots \cdot \underbrace{\int_R exp\left[-\frac{1}{2h^2}\left(t_{k-1} - x_i^{k-1}\right)^2\right] dt_{k-1}}_{=I} \cdot \underbrace{\int_R exp\left[-\frac{1}{2h^2}\left(t_{k+1} - x_i^{k+1}\right)^2\right] dt_{k+1}}_{=I} \cdot \dots \cdot$$
$$\underbrace{\int_R exp\left[-\frac{1}{2h^2}\left(t_d - x_i^d\right)^2\right] dt_d}_{=I} \Bigg\} \Bigg) \qquad (3.1)$$

In the above expression it is seen that any of the **d – 1** integrals has the same value with the others. Let's mark it with *I*. Calculating the value of *I*, we found that: $I = h\sqrt{2\pi}$

Replacing the calculated value of *I* above, results that:

$$\hat{f}_{\{k\}}(t_k) = \frac{1}{nh^d (2\pi)^{d/2}} \cdot h^{d-1}\left(\sqrt{2\pi}\right)^{d-1} \cdot \sum_{i=1}^{n} exp\left[-\frac{1}{2h^2}\left(t_k - x_i^k\right)^2\right]$$

or:

$$\hat{f}_{\{k\}}(t_k) = \frac{1}{nh\sqrt{2\pi}} \cdot \sum_{i=1}^{n} exp\left[-\frac{1}{2h^2}\left(t_k - x_i^k\right)^2\right] \qquad (3.2)$$

Now, replacing the expression of $\hat{f}_{(k)}(t_k)$ at the expression of the conditional density (*), we have:

$$\hat{f}\left(\underline{t}/t_k\right) = \frac{\hat{f}(\underline{t})}{\hat{f}_{\{k\}}(t_k)} = \frac{1}{h^{d-1}(2\pi)^{\frac{d-1}{2}}} \cdot \sum_{i=1}^{n} \frac{\overbrace{\left\{exp\left[-\frac{1}{2h^2}(t_1 - x_i^1)^2\right] \cdot \dots \cdot exp\left[-\frac{1}{2h^2}(t_k - x_i^k)^2\right] \cdot \dots \cdot exp\left[-\frac{1}{2h^2}(t_d - x_i^d)^2\right]\right\}}^{d\ factors}}{\sum_{i=1}^{n} exp\left[-\frac{1}{2h^2}(t_k - x_i^k)^2\right]}$$

(3.3)

Transforming the denominator, we get the final expression of the *conditional density estimate* with Gaussian Kernel, with 1 dimensional condition:

$$\hat{f}\left(\underline{t}/t_k\right) = \frac{1}{h^{d-1}(2\pi)^{\frac{d-1}{2}}} \cdot \frac{\sum_{i=1}^{n} exp\left\{-\frac{1}{2h^2}\overbrace{\left[\left(t_1-x_i^1\right)^2+\dots+\left(t_k-x_i^k\right)^2+\dots+\left(t_d-x_i^d\right)^2\right]}^{d\ monoms\ (terms)}\right\}}{\sum_{i=1}^{n} exp\left[-\frac{1}{2h^2}\left(t_k-x_i^k\right)^2\right]} \tag{3.4}$$

Theorem 2: *The conditional density estimate with Gaussian Kernel, with m dimensional condition is:*

$$\hat{f}\left(\underline{t}/t_{k+1},\dots,t_{k+m}\right) =$$

$$= \frac{1}{h^{d-m}(2\pi)^{\frac{d-m}{2}}} \cdot \frac{\sum_{i=1}^{n} exp\left\{-\frac{1}{2h^2}\overbrace{\left[\left(t_1-x_i^1\right)^2+\ \dots\ +\left(t_d-x_i^d\right)^2\right]}^{d\ monomials}\right\}}{\sum_{i=1}^{n} exp\left\{-\frac{1}{2h^2}\underbrace{\left[\left(t_{k+1}-x_i^{k+1}\right)^2+\ \dots\ +\left(t_{k+m}-x_i^{k+m}\right)^2\right]}_{m\ monomials}\right\}}$$

Proof:

Similarly with 1 dimensional case, conditional density estimate (*) is written as follows. The parameters are: *k, m, k + m* ∈ {1,2,…, d}, where *1 ≤ m ≤ d – 1*.The variables of the condition are $t_{k+1},\dots,t_{k+m}$.

$$\hat{f}\left(\underline{t}/t_{k+1},\dots,t_{k+m}\right) = \frac{\hat{f}(\underline{t})}{\hat{f}_{\{k+1,\dots,k+m\}}(t_{k+1},\dots,t_{k+m})} \tag{4}$$

The denominator, marginal density estimate of the components *(k+1, …,k+m)*, is (similarly with 1 dimensional case):

$$\hat{f}_{\{k+1,\dots,k+m\}}(t_{k+1},\dots,t_{k+m}) =$$

$$\frac{1}{nh^m(2\pi)^{m/2}} \cdot \sum_{i=1}^{n} \underbrace{\left\{exp\left[-\frac{1}{2h^2}(t_{k+1}-x_i^{k+1})^2\right]\cdot\ \dots \cdot exp\left[-\frac{1}{2h^2}(t_{k+m}-x_i^{k+m})^2\right]\right\}}_{m\ factors} \tag{4.1}$$

Transforming the expression under the sum sign, we get:

$$\hat{f}_{\{k+1,\dots,k+m\}}(t_{k+1},\dots,t_{k+m}) = \frac{1}{nh^m(2\pi)^{m/2}} \cdot \sum_{i=1}^{n} exp\left\{-\frac{1}{2h^2}\underbrace{\left[\left(t_{k+1}-x_i^{k+1}\right)^2 + \dots + \left(t_{k+m}-x_i^{k+m}\right)^2\right]}_{m\ terms}\right\} \quad (4.2)$$

Then, replacing the above expression at (4), the conditional density estimate is written as follows:

$$\hat{f}\left(\underline{t}/t_{k+1},\dots,t_{k+m}\right) = \frac{1}{h^{d-m}(2\pi)^{\frac{d-m}{2}}} \cdot \frac{\sum_{i=1}^{n} \overbrace{\left\{exp\left[-\frac{1}{2h^2}(t_1-x_i^1)^2\right] \cdot \dots \cdot exp\left[-\frac{1}{2h^2}(t_d-x_i^d)^2\right]\right\}}^{d\ factors}}{\sum_{i=1}^{n} \underbrace{\left\{exp\left[-\frac{1}{2h^2}(t_{k+1}-x_i^{k+1})^2\right] \cdot \dots \cdot exp\left[-\frac{1}{2h^2}(t_{k+m}-x_i^{k+m})^2\right]\right\}}_{m\ factors}} \quad (4.3)$$

Transforming the numerator and denominator of the above expression, we get the final expression of the *conditional density estimate* with Gaussian Kernel, with **m** dimensional condition:

$$\hat{f}\left(\underline{t}/t_{k+1},\dots,t_{k+m}\right) = \frac{1}{h^{d-m}(2\pi)^{\frac{d-m}{2}}} \cdot \frac{\sum_{i=1}^{n} exp\left\{-\frac{1}{2h^2}\overbrace{\left[(t_1-x_i^1)^2 + \dots + (t_d-x_i^d)^2\right]}^{d\ monomials}\right\}}{\sum_{i=1}^{n} exp\left\{-\frac{1}{2h^2}\underbrace{\left[(t_{k+1}-x_i^{k+1})^2 + \dots + (t_{k+m}-x_i^{k+m})^2\right]}_{m\ monomials}\right\}} \quad (4.4)$$

Theorem 3: *The conditional density estimate with Epanechnikov Kernel, with 1 dimensional condition is:*

$$\hat{f}\left(\underline{t}/t_k\right) = \frac{\hat{f}(\underline{t})}{\hat{f}_{\{k\}}(t_k)} = \frac{\Gamma\left(1+\frac{d-1}{2}\right)}{h^{d-1}\cdot\pi^{\frac{d-1}{2}}} \cdot \frac{\sum_{i=1}^{n} \overbrace{\left\{h^2 - \underbrace{[(t_1-x_i^1)^2 + (t_2-x_i^2)^2 + \dots + (t_d-x_i^d)^2]}_{d\ monomials}\right\}}^{for\ (t_1,t_2,\dots,t_d)\in\Delta_i;\ else\ 0}}{\frac{2nh^2}{d+1} - \sum_{i=1}^{n}\left(t_k - x_i^k\right)^2}$$

Proof:

As is mentioned in Tanku and Ceca (2013), eq. 4.2.4., density estimates with Epanechnikov kernel is given as follows:

$$\hat{f}(\underline{t}) = \frac{1}{nh^d} \cdot \frac{d+2}{2C_d} \cdot \sum_{i=1}^{n} a_i,$$

$$\text{where } a_i = \begin{cases} \left\{1 - \frac{1}{h^2}[(t_1 - x_i^1)^2 + (t_2 - x_i^2)^2 + \cdots + (t_d - x_i^d)^2]\right\}, & \{\ldots\} > 0 \\ 0 & , \{\ldots\} \le 0 \end{cases},$$

and C_d is the volume of sphere of unit radius in R^d.

Assign Δ_i the zone of nonnegative terms a_i given above. With a simple transformation, the zone Δ_i is given:

$$\Delta_i = \{(t_1, t_2, \ldots, t_d) \in R^d \big| \{h^2 - [(t_1 - x_i^1)^2 + (t_2 - x_i^2)^2 + \cdots + (t_d - x_i^d)^2]\} > 0\}$$

Replacing it above at the expression of the density estimate, we get the analytical form of nonnegative values of this density, for $(t_1, t_2, \ldots t_d) \in R^d$:

$$\hat{f}(\underline{t}) = \frac{1}{nh^d} \cdot \frac{d+2}{2C_d} \cdot \sum_{i=1}^{n} \overbrace{\left\{1 - \frac{1}{h^2}[(t_1 - x_i^1)^2 + (t_2 - x_i^2)^2 + \cdots + (t_d - x_i^d)^2]\right\}}^{for\ (t_1, t_2, \ldots, t_d) \in \Delta_i;\ else\ 0} =$$

$$= \frac{1}{nh^{d-2}} \cdot \frac{d+2}{2C_d} \cdot \sum_{i=1}^{n} \overbrace{\left\{h^2 - [(t_1 - x_i^1)^2 + (t_2 - x_i^2)^2 + \cdots + (t_d - x_i^d)^2]\right\}}^{for\ (t_1, t_2, \ldots, t_d) \in \Delta_i;\ else\ 0}$$

Let's calculate the denominator at (3), means marginal density estimates , which is function only of the variable t_k and is integrated by all other variables $t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_d$.

$$\hat{f}_{\{k\}}(t_k) = \int_{R^{d-1}} \hat{f}(\underline{t}) \cdot dt_1 \ldots dt_{k-1} dt_{k+1} \ldots dt_d ==$$

$$\frac{1}{nh^{d-2}} \cdot \frac{d+2}{2C_d} \cdot \sum_{i=1}^{n} \underbrace{\int \ldots \int}_{\substack{(d-1)\ dim; \\ \Delta_i}} \overbrace{\{h^2 - [(t_1 - x_i^1)^2 + (t_2 - x_i^2)^2 + \cdots + (t_d - x_i^d)^2]\} \cdot dt_1 \ldots dt_{k-1} dt_{k+1} \ldots dt_d}^{IE} \quad (5)$$

We called the integral under the sign of the sum 1 *dimensional Epanechnikov Integral* – shortly ***IE***. The calculation of the integral is given in Annex 1. Meanwhile, the result is given below:

$$IE = \frac{2h^{d+1}}{d+1} \cdot \frac{\pi^{\frac{d-1}{2}}}{\Gamma\left(1+\frac{d-1}{2}\right)} - \left(t_k - x_i^k\right)^2 \cdot \frac{\pi^{\frac{d-1}{2}} \cdot h^{d-1}}{\Gamma\left(1+\frac{d-1}{2}\right)} \qquad (5.1)$$

Replacing at the marginal density estimates $\hat{f}_{(k)}(t_k)$ at eq.5 above, we get:

$$\hat{f}_{\{k\}}(t_k) = \frac{1}{nh^{d-2}} \cdot \frac{d+2}{2C_d} \cdot \sum_{i=1}^{n} \left(\frac{2h^{d+1}}{d+1} \cdot \frac{\pi^{\frac{d-1}{2}}}{\Gamma\left(1+\frac{d-1}{2}\right)} - \left(t_k - x_i^k\right)^2 \cdot \frac{\pi^{\frac{d-1}{2}} \cdot h^{d-1}}{\Gamma\left(1+\frac{d-1}{2}\right)} \right) \qquad (5.2)$$

Factorizing 5.2 above:

$$\hat{f}_{\{k\}}(t_k) = \frac{d+2}{2nC_d} \cdot \frac{h \cdot \pi^{\frac{d-1}{2}}}{\Gamma\left(1+\frac{d-1}{2}\right)} \cdot \left\{ \frac{2nh^2}{d+1} - \sum_{i=1}^{n} \left(t_k - x_i^k\right)^2 \right\} \qquad (5.3)$$

Replacing the above expression of the marginal density estimates $\hat{f}_{(k)}(t_k)$ at the expression of conditional density estimates, we get as follows:

$$\hat{f}\left(\frac{t}{t_k}\right) = \frac{\hat{f}(t)}{\hat{f}_{\{k\}}(t_k)} = \frac{\frac{1}{nh^{d-2}} \cdot \frac{d+2}{2C_d} \cdot \sum_{i=1}^{n} \overbrace{\left\{h^2 - \left[\left(t_1 - x_i^1\right)^2 + \left(t_2 - x_i^2\right)^2 + \cdots + \left(t_d - x_i^d\right)^2\right]\right\}}^{for\ (t_1, t_2, \ldots, t_d) \in \Delta_i;\ else\ 0}}{\frac{1}{nh^{d-2}} \cdot \frac{d+2}{2C_d} \cdot \sum_{i=1}^{n} \left(\frac{2h^{d+1}}{d+1} \cdot \frac{\pi^{\frac{d-1}{2}}}{\Gamma\left(1+\frac{d-1}{2}\right)} - \left(t_k - x_i^k\right)^2 \cdot \frac{\pi^{\frac{d-1}{2}} \cdot h^{d-1}}{\Gamma\left(1+\frac{d-1}{2}\right)} \right)} \qquad (5.4)$$

And transforming the expression 5.4 above, we get the final form of the *conditional density estimate with Epanechnikov Kernel with 1 dimensional condition:*

$$\hat{f}\left(\frac{t}{t_k}\right) = \frac{\hat{f}(t)}{\hat{f}_{\{k\}}(t_k)} = \frac{\Gamma\left(1+\frac{d-1}{2}\right)}{h^{d-1} \cdot \pi^{\frac{d-1}{2}}} \cdot \frac{\sum_{i=1}^{n} \overbrace{\left\{h^2 - \underbrace{\left[\left(t_1 - x_i^1\right)^2 + \left(t_2 - x_i^2\right)^2 + \cdots + \left(t_d - x_i^d\right)^2\right]}_{d\ monomials}\right\}}^{for\ (t_1, t_2, \ldots, t_d) \in \Delta_i;\ else\ 0}}{\frac{2nh^2}{d+1} - \sum_{i=1}^{n} \left(t_k - x_i^k\right)^2} \qquad (5.5)$$

Theorem 4: The conditional density estimate with Epanechnikov Kernel, with m dimensional condition is:

$$\hat{f}\left(\underline{t}/_{t_{k+1},\ldots,t_{k+m}}\right) = \frac{\Gamma\left(1+\frac{d-m}{2}\right)}{h^{d-m}\cdot\pi^{\frac{d-m}{2}}} \cdot \sum_{i=1}^{n} \frac{\overbrace{\left\{h^2-\underbrace{\left[\left(t_1-x_i^1\right)^2+\left(t_2-x_i^2\right)^2+\cdots+\left(t_d-x_i^d\right)^2\right]}_{d\ monomials}\right\}}^{for\left(t_1,t_2,\ldots,t_d\right)\in\Delta_i;\ else\,0}}{\frac{2nh^2}{d-m+2}-\sum_{i=1}^{n}\left[\left(t_{k+1}-x_i^{k+1}\right)^2+\cdots+\left(t_{k+m}-x_i^{k+m}\right)^2\right]}\Bigg]\Bigg]$$

Proof:

The general form of conditional density estimate, with *m* dimensional condition (4), is written below. Parameters are: $k, m, k+m \in \{1,2,\ldots,d\}$ and $t_{k+1}, \ldots, t_{k+m}$ the variables of the condition.

$$\hat{f}\left(\underline{t}/_{t_{k+1},\ldots,t_{k+m}}\right) = \frac{\hat{f}(\underline{t})}{\hat{f}_{\{k+1,\ldots,k+m\}}(t_{k+1},\ldots,t_{k+m})}$$

The denominator, marginal density estimate of the components *(k+1, …,k+m)*, is (similarly with 1 dimensional case):

$$\hat{f}_{\{k+1,\ldots,k+m\}}(t_{k+1},\ldots,t_{k+m}) = \int_{R^{d-m}} \hat{f}(\underline{t})\cdot dt_1 \ldots dt_k\, dt_{k+m+1}\ldots dt_d =$$

$$\frac{1}{nh^{d-2}}\cdot\frac{d+2}{2C_d}\cdot\sum_{i=1}^{n}\underbrace{\int\ldots\int}_{\substack{(d-m)\ dim;\\ \Delta_i}}\{h^2-[(t_1-x_i^1)^2+\cdots+(t_d-x_i^d)^2]\}\cdot dt_1\ldots dt_k dt_{k+m+1}\ldots dt_d \qquad (6)$$

Where Δ_i is like in previous cases:

$$\Delta_i = \left\{(t_1, t_2, \ldots, t_d) \in R^d \middle| \{h^2-[(t_1-x_i^1)^2+(t_2-x_i^2)^2+\cdots+(t_d-x_i^d)^2]\} > 0\right\} \qquad (6.1)$$

We called the integral under the sign of the sum m dimensional Epanechnikov Integral – shortly IE_m. The calculation of the integral is given in Annex 1. Meanwhile, the result is given below:

$$IE_m = \frac{2h^{d-m+2}}{d-m+2} \cdot \frac{\pi^{\frac{d-m}{2}}}{\Gamma\left(1+\frac{d-m}{2}\right)} - \left[\left(t_{k+1} - x_i^{k+1}\right)^2 + \ \dots + \left(t_{k+m} - x_i^{k+m}\right)^2\right] \cdot \frac{\pi^{\frac{d-m}{2}} \cdot h^{d-m}}{\Gamma\left(1+\frac{d-m}{2}\right)} \quad (6.2)$$

Replacing at the marginal density estimates $\hat{f}_{\{k+1,\dots,k+m\}}(t_{k+1}, \dots, t_{k+m})$ in equation (4) above, we get:

$$\hat{f}_{\{k+1,\dots,k+m\}}(t_{k+1}, \dots, t_{k+m}) =$$

$$= \frac{1}{nh^{d-2}} \cdot \frac{d+2}{2C_d} \cdot \cdot \sum_{i=1}^{n} \left(\frac{2h^{d-m+2}}{d-m+2} \cdot \frac{\pi^{\frac{d-m}{2}}}{\Gamma\left(1+\frac{d-m}{2}\right)} - \left[\left(t_{k+1} - x_i^{k+1}\right)^2 + \dots + \left(t_{k+m} - x_i^{k+m}\right)^2\right] \cdot \frac{\pi^{\frac{d-m}{2}} \cdot h^{d-m}}{\Gamma\left(1+\frac{d-m}{2}\right)} \right) \quad (6.3)$$

It is not difficult to see that the formula (5-5) of marginal density estimates for the case m = 1, is a special case of the 6.3 above formula.

Factorizing above:

$$\hat{f}_{\{k+1,\dots,k+m\}}(t_{k+1}, \dots, t_{k+m}) =$$

$$= \frac{d+2}{2nC_d} \cdot \frac{h^{2-m} \cdot \pi^{\frac{d-m}{2}}}{\Gamma\left(1+\frac{d-m}{2}\right)} \cdot \left\{ \frac{2nh^2}{d-m+2} - \sum_{i=1}^{n} \left[\left(t_{k+1} - x_i^{k+1}\right)^2 + \dots + \left(t_{k+m} - x_i^{k+m}\right)^2\right] \right\} \quad (6.4)$$

Replacing the above expression of the marginal density estimates $\hat{f}_{\{k+1,\dots,k+m\}}(t_{k+1}, \dots, t_{k+m})$ at the expression of conditional density estimates (4), we get as follows:

$$\hat{f}\left(\underline{t}/t_{k+1}, \dots, t_{k+m}\right) = \frac{\hat{f}(\underline{t})}{\hat{f}_{\{k+1,\dots,k+m\}}(t_{k+1}, \dots, t_{k+m})} =$$

$$= \frac{\frac{1}{nh^{d-2}} \cdot \frac{d+2}{2C_d} \cdot \sum_{i=1}^{n} \overbrace{\left\{h^2 - \left[\left(t_1 - x_i^1\right)^2 + \left(t_2 - x_i^2\right)^2 + \dots + \left(t_d - x_i^d\right)^2\right]\right\}}^{for\ (t_1, t_2, \dots, t_d) \in \Delta_i;\ else\ 0}}{\frac{d+2}{2nC_d} \cdot \frac{h^{2-m} \cdot \pi^{\frac{d-m}{2}}}{\Gamma\left(1+\frac{d-m}{2}\right)} \cdot \left\{ \frac{2nh^2}{d-m+2} - \sum_{i=1}^{n} \left[\left(t_{k+1} - x_i^{k+1}\right)^2 + \dots + \left(t_{k+m} - x_i^{k+m}\right)^2\right] \right\}} \quad (6.5)$$

And transforming the expression above, we get the final form *of the conditional density estimate with Epanechnikov Kernel with* **m** *dimensional condition:*

$$\hat{f}\left(\underline{t}/t_{k+1},\dots,t_{k+m}\right)=\frac{\Gamma\left(1+\frac{d-m}{2}\right)}{h^{d-m}\cdot\pi^{\frac{d-m}{2}}}\cdot\frac{\sum_{i=1}^{n}\overbrace{\left\{h^2-\underbrace{\left[\left(t_1-x_i^1\right)^2+\left(t_2-x_i^2\right)^2+\dots+\left(t_d-x_i^d\right)^2\right]}_{d\ monomials}\right\}}^{for\ (t_1,t_2,\dots,t_d)\in\Delta_i;\ else\ 0}}{\frac{2nh^2}{d-m+2}-\sum_{i=1}^{n}\left[\left(t_{k+1}-x_i^{k+1}\right)^2+\dots+\left(t_{k+m}-x_i^{k+m}\right)^2\right]}$$

(6.6)

The results of the densities, marginal densities and conditional densities are summarized in the tables 1-3 respectively.

Table 1 The density estimates with Gaussian and Epanechnikov kernels[1]

$\hat{f}(\underline{t})=$	Gaussian kernel	$\hat{f}(\underline{t})=\frac{1}{nh^d(2\pi)^{d/2}}\cdot\sum_{i=1}^{n}\left\{exp\left[-\frac{1}{2h^2}(t_1-x_i^1)^2\right]\cdot\dots\cdot exp\left[-\frac{1}{2h^2}(t_k-x_i^k)^2\right]\cdot\dots\cdot exp\left[-\frac{1}{2h^2}(t_d-x_i^d)^2\right]\right\}$
	Epanechnikov kernel	$\hat{f}(\underline{t})=\frac{1}{nh^{d-2}}\cdot\frac{d+2}{2C_d}\cdot\sum_{i=1}^{n}\overbrace{\{h^2-[(t_1-x_i^1)^2+(t_2-x_i^2)^2+\dots+(t_d-x_i^d)^2]\}}^{for\ (t_1,t_2,\dots,t_d)\in\Delta_i;\ else\ 0}$ $ku\ \Delta_i=\left\{(t_1,t_2,\dots,t_d)\in R^d \middle| \left\{h^2-\left[(t_1-x_i^1)^2+(t_2-x_i^2)^2+\dots+\left(t_d-x_i^d\right)^2\right]\right\}>0\right\}$

[1] See "Density estimation for economic variables – a genuine application" Altin Tanku & Kliti Ceca, forthcoming, Bank of Albania Working Papers Series 2013, 08 (47) 2013.

Table 2 The marginal density estimates, 1 and m dimensional cases, for density estimates with Gaussian and Epanechnikov kernels

1 dimensional	$\hat{f}_{\{k\}}(t_k) =$	**Gaussian kernel**	$\frac{1}{nh\sqrt{2\pi}} \cdot \sum_{i=1}^{n} exp\left[-\frac{1}{2h^2}(t_k - x_i^k)^2\right]$
		Epanechnikov kernel	$\frac{d+2}{2nC_d} \cdot \frac{h \cdot \pi^{\frac{d-1}{2}}}{\Gamma\left(1+\frac{d-1}{2}\right)} \cdot \left\{\frac{2nh^2}{d+1} - \sum_{i=1}^{n}(t_k - x_i^k)^2\right\}$
m dimensional	$\hat{f}_{\{k+1,\dots,k+m\}}(t_{k+1},\dots,t_{k+m}) =$	**Gaussian kernel**	$\frac{1}{nh^m(2\pi)^{m/2}} \cdot \sum_{i=1}^{n} exp\left\{-\frac{1}{2h^2}\underbrace{\left[(t_{k+1} - x_i^{k+1})^2 + \dots + (t_{k+m} - x_i^{k+m})^2\right]}_{m\ monomials}\right\}$
		Epanechnikov kernel	$\frac{d+2}{2nC_d} \cdot \frac{h^{2-m} \cdot \pi^{\frac{d-m}{2}}}{\Gamma\left(1+\frac{d-m}{2}\right)} \cdot \left\{\frac{2nh^2}{d-m+2} - \sum_{i=1}^{n}\underbrace{\left[(t_{k+1} - x_i^{k+1})^2 + \dots + (t_{k+m} - x_i^{k+m})^2\right]}_{m\ monomials}\right\}$

Table 3 The conditional density estimates, 1 and m dimensional cases, for density estimates with Gaussian and Epanechnikov kernels.

Prerja 1 dimensonale	$f\left(\frac{t}{t_k}\right)=$	Gaussian kernel	$\frac{1}{h^{d-1}(2\pi)^{\frac{d-1}{2}}}\cdot\frac{\sum_{i=1}^{n} exp\left\{-\frac{1}{2h^2}\overbrace{[(t_1-x_i^1)^2+\ \dots\ +(t_k-x_i^k)^2+\ \dots\ +(t_d-x_i^d)^2]}^{d\ monomials}\right\}}{\sum_{i=1}^{n} exp\left[-\frac{1}{2h^2}(t_k-x_i^k)^2\right]}$
		Epanechnikov kernel	$\frac{\Gamma\left(1+\frac{d-1}{2}\right)}{h^{d-1}\cdot\pi^{\frac{d-1}{2}}}\cdot\frac{\sum_{i=1}^{n}\overbrace{\left\{h^2-\underbrace{[(t_1-x_i^1)^2+(t_2-x_i^2)^2+\cdots+(t_d-x_i^d)^2]}_{d\ monomials}\right\}}^{for\ (t_1,t_2,\dots,t_d)\in\Delta_i;\ else\ 0}}{\frac{2nh^2}{d+1}-\sum_{i=1}^{n}(t_k-x_i^k)^2}$
Prerja m dimensionale	$f\left(\frac{t}{t_{k+1},\dots,t_{k+m}}\right)=$	Gaussian kernel	$\frac{1}{h^{d-m}(2\pi)^{\frac{d-m}{2}}}\cdot\frac{\sum_{i=1}^{n} exp\left\{-\frac{1}{2h^2}\overbrace{[(t_1-x_i^1)^2+\ \dots\ +(t_d-x_i^d)^2]}^{d\ monomials}\right\}}{\sum_{i=1}^{n} exp\left\{-\frac{1}{2h^2}\underbrace{[(t_{k+1}-x_i^{k+1})^2+\ \dots\ +(t_{k+m}-x_i^{k+m})^2]}_{m\ monomials}\right\}}$
		Epanechnikov kernel	$\frac{\Gamma\left(1+\frac{d-m}{2}\right)}{h^{d-m}\cdot\pi^{\frac{d-m}{2}}}\cdot\frac{\sum_{i=1}^{n}\overbrace{\left\{h^2-\underbrace{[(t_1-x_i^1)^2+(t_2-x_i^2)^2+\cdots+(t_d-x_i^d)^2]}_{d\ monomials}\right\}}^{for\ (t_1,t_2,\dots,t_d)\in\Delta_i;\ else\ 0}}{\frac{2nh^2}{d-m+2}-\sum_{i=1}^{n}[(t_{k+1}-x_i^{k+1})^2+\cdots+(t_{k+m}-x_i^{k+m})^2]}$

The above densities are the analytical expression of the cross section of the definition 1. They represent continuous density functions of the variable of interest (dependent variable) for all potential values of other d – 1 variables (independent variables). Definition 1 provides simultaneously the tool of investigation and the metric of interpretation of the relationships among our variables of interest. This last one due to the fact that any of the conditional probabilities (the 4 cases above in table 3) represents probability densities and their numerical characteristics can be used to measure and compare these densities among each-other. In the following chapter we would use this method (cross section) and the first and the second moment of the distribution to investigate the relationship between money and inflation in the case of US economy.

3. APPLICATION IN THE EMPIRICAL ANALYSIS OF THE RELATIONSHIP BETWEEN MONEY AND INFLATION IN THE CASE OF US ECONOMY

Here below we will try to expose and explore the use of this multidimensional density estimation and the cross section methodology as a tool of investigation of the relationship between money and inflation in the case of US economy for the period 1959-2013. The choice of US economy is due to the availability and the quality of the data.

In principal economists believe that inflation is a monetary phenomenon, meaning that inflation is a result of the money growth. From Stewart Mill to Irvin Fisher to Milton Friedman economists believe that it is the speed of money growth and the behavior of the central bank is the culprit behind the behavior of inflation in the economy. The idea is based on the famous equation of exchange below

$$M \times V = P \times Y$$

Where **M** presents the money stock; **V** the velocity of circulation or the number of times that money changes hands in the economy; P stands for the price level and Y is the real GDP.

Assuming that velocity is constant (as monetary theorists do) one would expect that changes in the growth rate of monetary aggregates will be the main cause behind changes in the growth rate of price level or inflation. Alternatively if prices remain constant too, we were to get faster growth rate and the economy will experience the phenomenon of money illusion. In other words money would cause a faster increase in output. If any of the above cases were true, money will cause inflation and output growth respectively. Meaning, that money will ganger cause either inflation or output growth. In line with this theoretic foundation we here try to investigate whether inflation is a monetary phenomenon or not.

In the particular scenario that monetary theory is proven correct one would expect that changes in money growth will be matched by changes in the same direction of inflation and or GDP. For this purpose we use a data set of quarterly observations of the nominal Money and CPI and real GDP. All data comes from FRED and cover the period 1959-Q1 – 2013-Q1. The figures are in reported in USD, for money stock and real GDP and in index form for CPI. To test whether money causes inflation or real GDP in addition to simultaneous observations of the variables of interest the dataset incorporates also 1,2,3,4,6, lagged series of money in order to check the granger causality problems. Like in Tanku and Ceca (2013), the proposal is to reorganize the common (time series concept) dataset in a new way, which will transform the database as follows:

$$\begin{cases} \underline{\mathrm{x}}_1 = \left(\mathrm{x}_1^1, \mathrm{x}_1^2, \ldots, \mathrm{x}_1^{\mathrm{d}}\right) \\ \underline{\mathrm{x}}_2 = \left(\mathrm{x}_2^1, \mathrm{x}_2^2, \ldots, \mathrm{x}_2^{\mathrm{d}}\right) \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\ \underline{\mathrm{x}}_\mathrm{n} = \left(\mathrm{x}_\mathrm{n}^1, \mathrm{x}_\mathrm{n}^2, \ldots, \mathrm{x}_\mathrm{n}^{\mathrm{d}}\right) \end{cases} \quad \textit{alternatively} \quad X_{1\times n} = \mathcal{F}_{d\times n}$$

x_i for the dependent variable is $i \in \{1, 2, \ldots, n\}$ and the rest of the dataset x_i^{d-1} represent independent variables at time lag =L, where

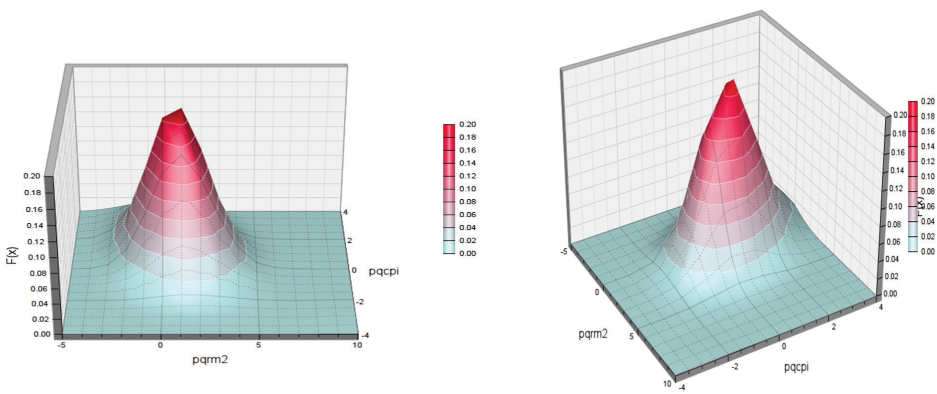
$i \in \{1 + L, \ \dots, n - L\}$ and $L \in \{1, 2, 3, 4, 6\}$. The idea is to create vectors

of data which represent simultaneous observations of all variables of interest (which together comprise show envision the state of what we call the economy at one moment in time) and count in a multi-dimensional cartesian system the number of jointly occurring events along each dimension. Now these simultaneous data set entries represent current Values at time t of CPI and DGP pared with lagged observations of money values of money at time *t-L*.

The results in the figure 1 show the result of joint PDF of the couple real money and CPI in percentage changes[2]. Figure 1 shows the three dimensional representation of the joint density function of money & CPI of the calculated joint probability density estimation and the corresponding contour (view from above) representation.[3] The shape and the position of the estimated density are analyzed to judge on the existence or inexistence of the relationship among these two variables Tanku and Ceca (2013).

FIG. 1 - JOINT DENSITY PROBABILITY FUNCTION RM2 VS. CPI (PERCENTAGE CHANGE, QUARTERLY OBSERVATIONS)

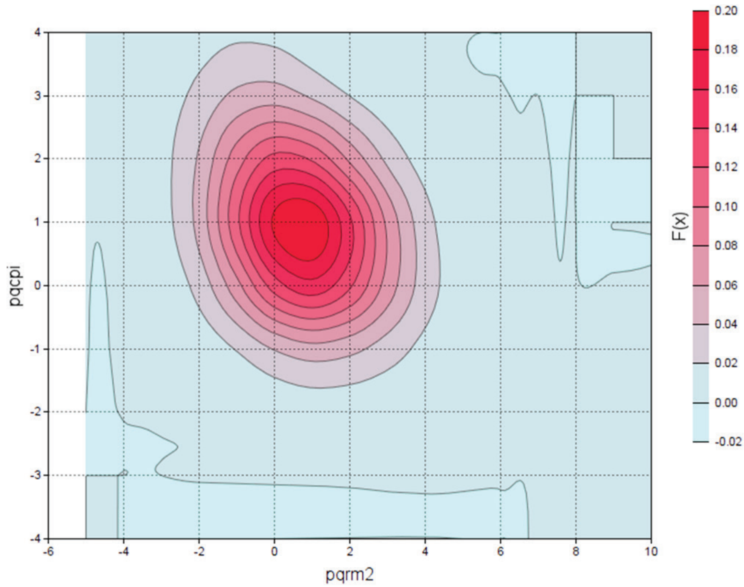
1.A. SURFACE REPRESENTATION



[2] Tanku and Ceca (2013) use the first difference data as an indication of the slope of the relationship and changes in percentage change as elasticity.

[3] For the reader the intensity of the color shows the value of the density function, with darker colors meaning higher densities.

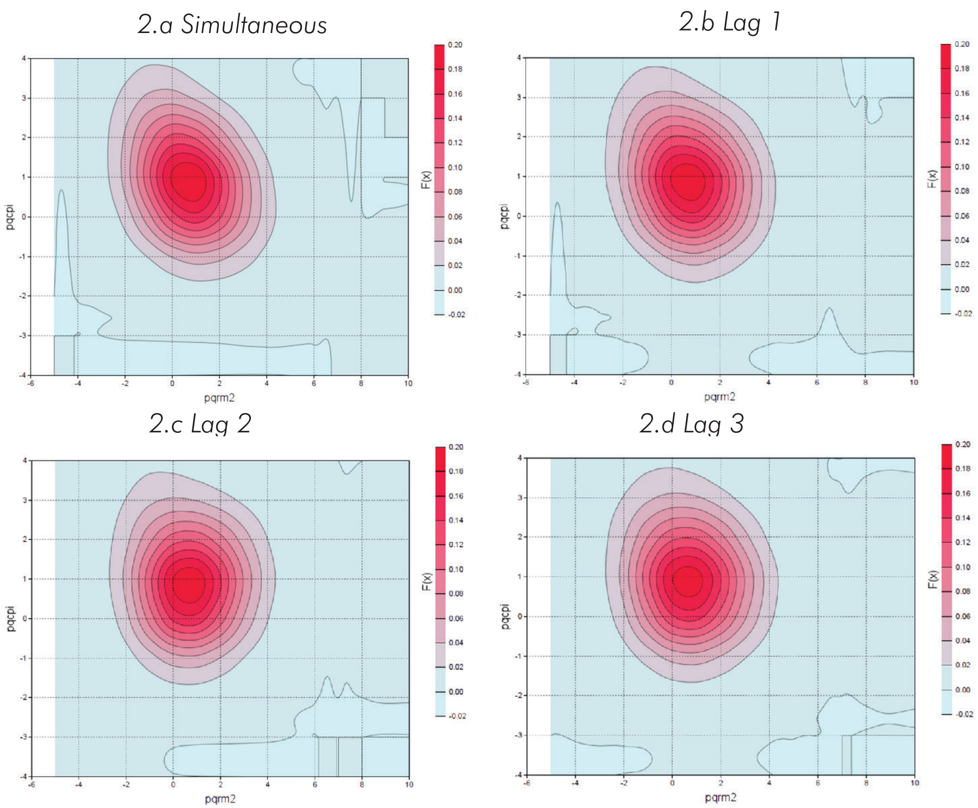
1.B. CONTOUR REPRESENTATION



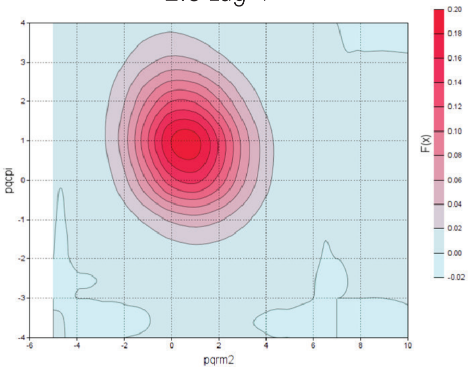
According to the interpretation given by them the shape of a perfect bell (or any symmetrically shaped object) which main axes are perpendicular to one or both axes is interpreted as a sign of independence between the variables of interests. An oval shaped bell (or any other irregularly shaped Joint probability distribution) which main diagonal rests at an angle with the axes is interpreted as a sign that changes in one of the variables are meet by scales changes in the other variable implying a dependence on each other or a third variable which is not represented in the Graph. The same is true in the case when several symmetrically and diagonally perpendicular shaped objects spread at an angle in the three dimensional space will indicate dependence to each other or a third variable. Therefore the stronger the deviation the stronger is the response or the reply of the depended variable to changes in the independent one.

Therefore the first analysis I based on the shape of the JPDF, figure 2 presents the contours of the JPDF of inflation and money starting with simultaneous observation in fig. 2.a., with the rest of figure covering results for lags 1-4 and 6th respectively. The lag figures providing evidence of money granger causing inflation. At this point our analysis is focused on the shape and angle of the resulting figures along the lines described above. It currently seems that the fig 2.d has the largest deviation from the axes which we interpret as the 2nd lag exerting the strongest and the fullest effects of money in inflation. Therefore we will take this and apply the "cross section" analysis to further explore the money inflation relationship. Alternatively we could perform the cross section analysis on the all cases depicted in figure 2, however given our interest on the application rather than on the relationship, in that case we will constrain our analysis to figure 2.b.

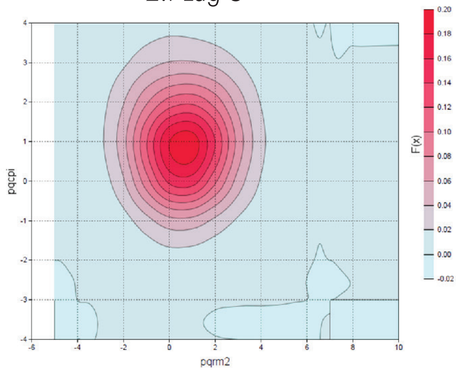
FIG.2. - JOINT DENSITY PROBABILITY FUNCTION RM2 VS. CPI (PERCENTAGE CHANGE, QUARTERLY OBSERVATIONS)



2.e Lag 4



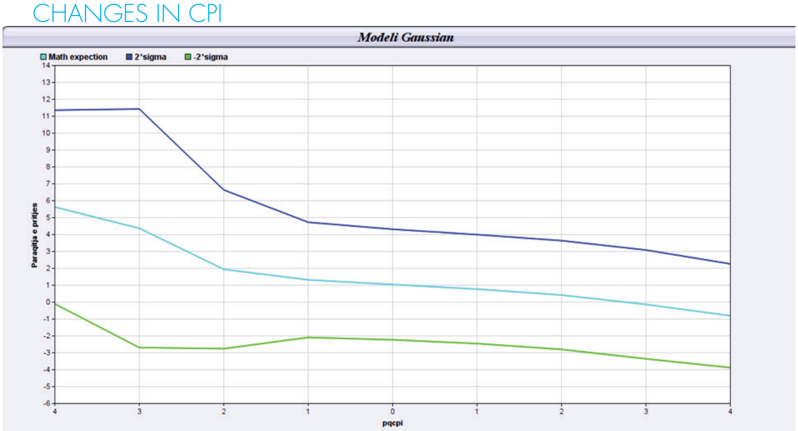
2.f Lag 5



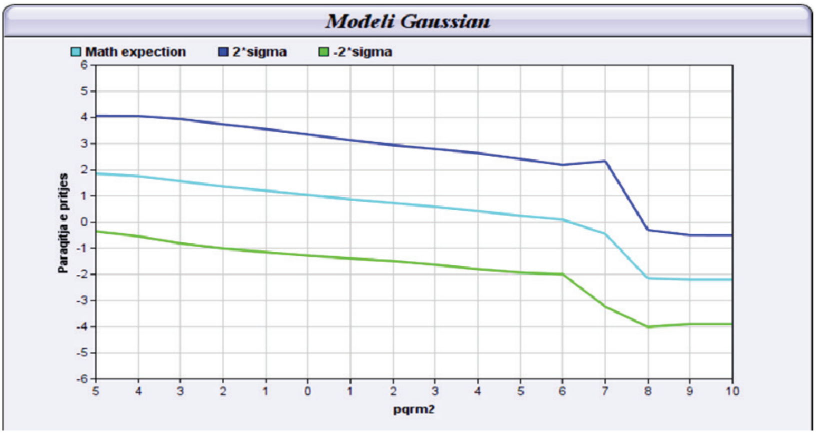
The result of the cross section analysis is represented in figure 3. What figure 3 shows are the first two moments of the density functions that have resulted from the cross sectional cuts of the JPDF presented in fig.2.b. figure 3.a. and 3.b. show the mathematic mean and the deviation of continuously calculated density functions of money (3.a.) and inflation (3.b.) plotted for all values of percentage changes in inflation and real money respectively, as described by equation given by the first row of table 3, above. The analysis could easily include all other numerical characteristics of the estimated densities. However the first two moments are good enough means of analysis since they show how the expected value of inflation and its uncertainty changes for different values of money growth.

FIG. 3 CROSS SECTION OF JOINT DENSITY PROBABILITY FUNCTION DRM2 VS. DCPI, CORRESPONDING TO FIG. 2.B.

3.A. EXPECTED VALUE & STANDARD DEVIATION OF REAL M2 FOR CHANGES IN CPI



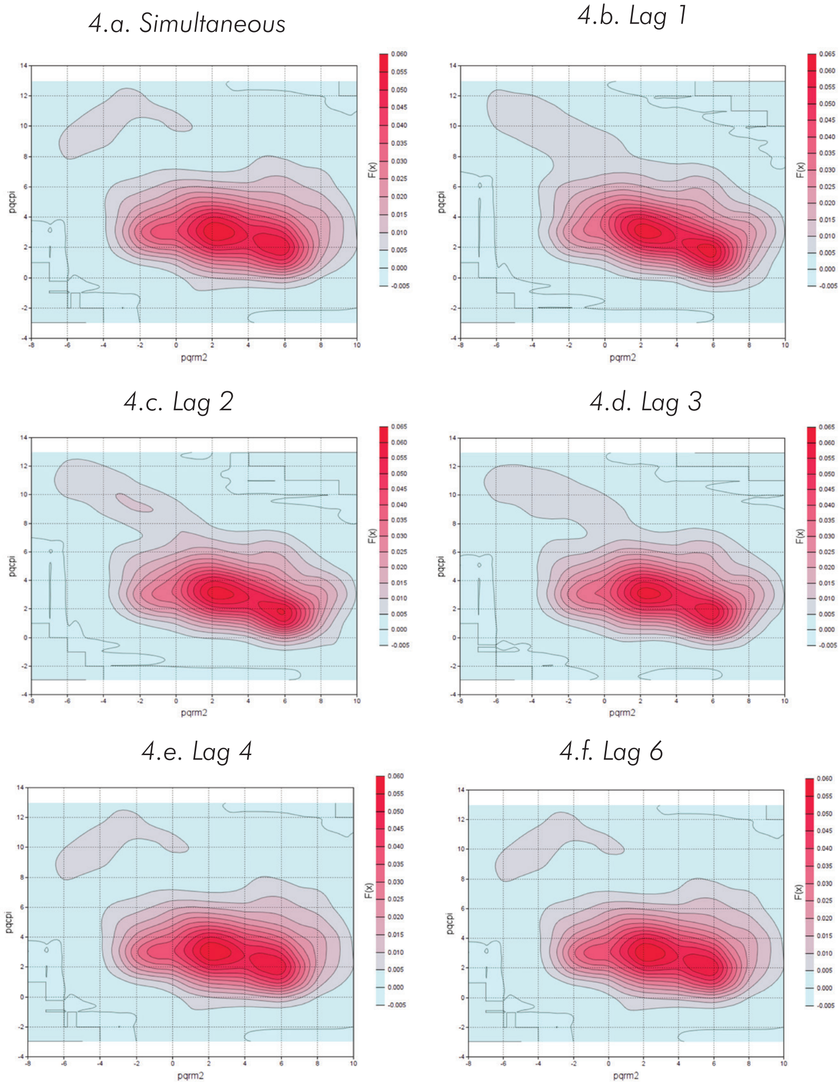
3.B. EXPECTED VALUE & STANDARD DEVIATION OF CPI FOR DIFFERENT CHANGES IN REAL M2



Since we expect that past values of money growth influence inflation we analyze developments in fig. 3.b.[4] From the graph is clear that inflation respond to changes in the rate of monetary growth. The downward sloping curve indicates a sluggish and linear response of inflation to changes of money until the growth rate reaches 6 percent. The response becomes much sharper when money growth reaches between 6 and 8 percent. Beyond 8 percent money growth, inflation does not seem to be affected by changes in money supply. However fig. 3.b., shows that in general increase in money growth is associated with lower inflation. This result is inconsistent with monetary theory and the belief that inflation is a monetary phenomenon. Our analysis shows that data do not support this theory.

[4] Fig. 3.a. shows the behavior of money growth for different values of inflation.

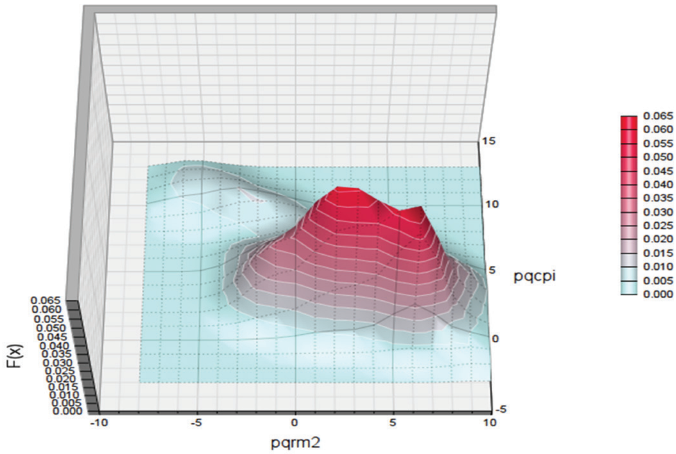
FIG. 4 - JOINT DENSITY PROBABILITY FUNCTION RM2 VS. CPI
(PERCENTAGE CHANGE, ANNUAL OBSERVATIONS)



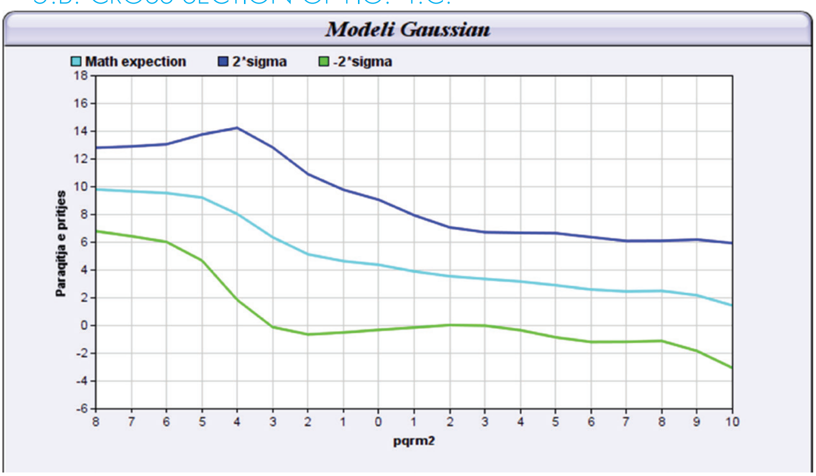
We have repeated the same exercise on annual observations (annual percentage changes of money and CPI), and the results are presented in figure 4. From fig. 4 it is clear that lag = 2 has the strongest deviation from the axes and the effect wears off in the following lags. This we interpret that the annual growth of money exerts its strongest effect in inflation after two lags and therefore we have chosen to provide the cross section analysis for the second lag.

FIG. 5 - REAL MONEY AND INFLATION, ANNUAL OBSERVATIONS, LAG 2

5.A. SURFACE REPRESENTATION OF FIG. 4.C.



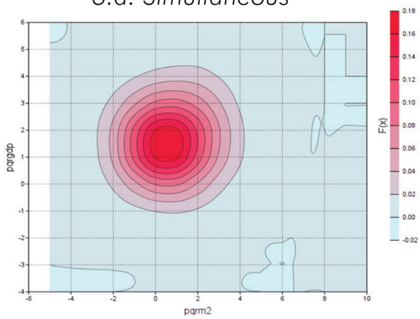
5.B. CROSS SECTION OF FIG. 4.C.



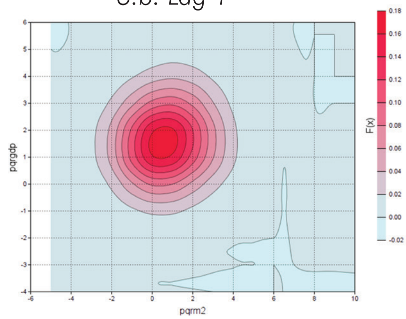
Like in the case of quarterly observations figure 5.b. shows that inflation and the growth rate of money are connected in a negative relationship. The response of inflation is stronger for money growth rates increase from -5 to -2 and from 8 to 10 percent annually. Despite that, the negative relationship does not support the monetarist's theory.

FIG. 6 - JOINT DENSITY PROBABILITY FUNCTION RM2 VS. RGDP

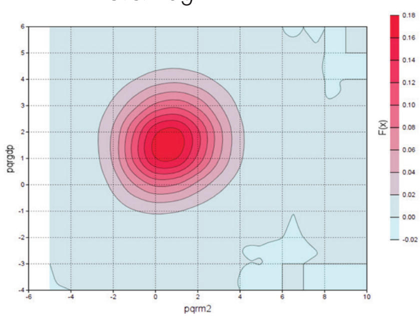
6.a. Simultaneous



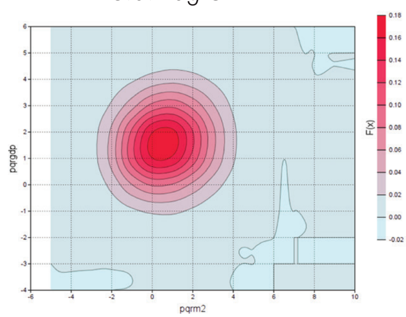
6.b. Lag 1



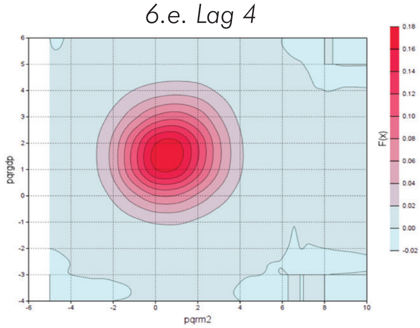
6.c. Lag 2



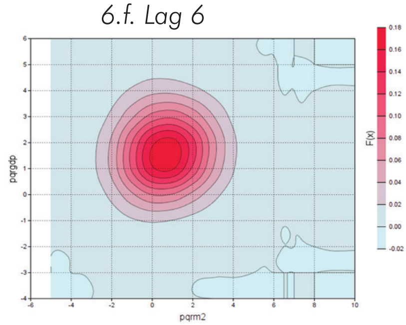
6.d. Lag 3



6.e. Lag 4



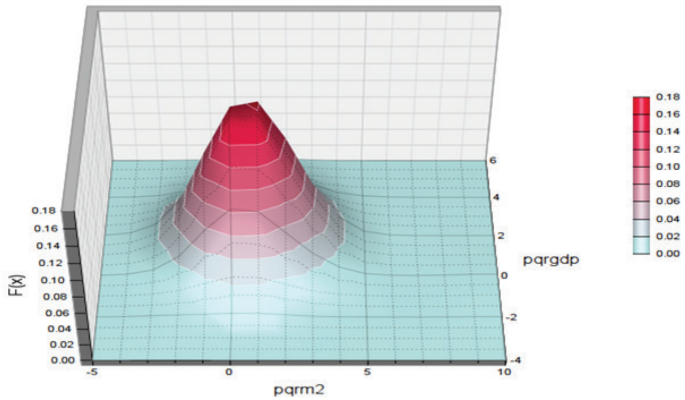
6.f. Lag 6



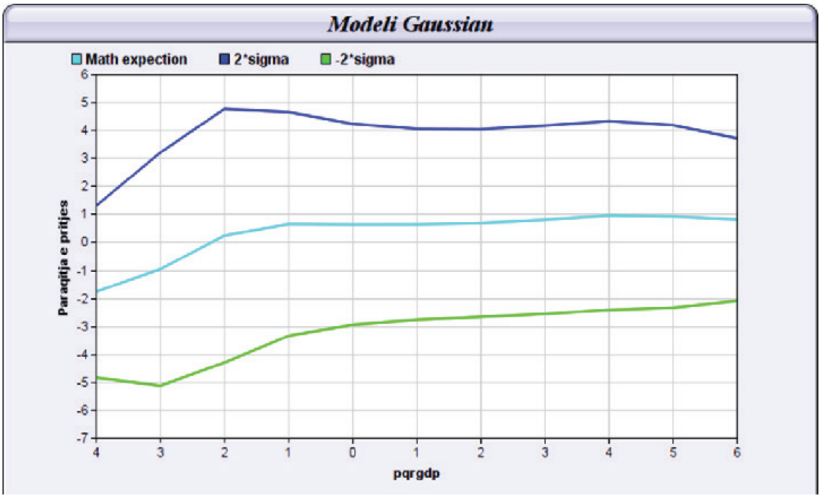
In a final attempt we try to investigate whether quarterly changes in money growth are mirrored by growth rates of real GDP. The results of this exercise are shown in figure 6. They indicate almost perfectly shaped bells which indicate that variables are independent from each-other. However it seems that panel b of figure 6 seems to provide hints of a potential relationship between real money and real GDP. We therefore proceed with cross section to investigate further if the expected value of real GDP and its variance change in response to changes in money growth. The results are presented in figure 7. Interestingly figure 6.b. indicates that there is a positive and strong response (almost 1 to 1) in real GDP when money growth increases from -4 to -1 percent, but beyond this point the expected value and of real GDP remains almost constant despite changes in money growth. In the meantime standard deviation in GDP growth gets marginally smaller for money growth rates beyond 5 percent. Over all the relationship between money and growth does not add significant information on the empirical relevance of the equation of exchange.

FIG. 7 – JOINT DENSITY PROBABILITY FUNCTION RM2 VS. RGDP (PERCENTAGE CHANGE, QUARTERLY OBSERVATIONS)

7.A. SURFACE REPRESENTATION FIG. 6(5).B.



7.B. CROSS SECTION REPRESENTATION FIG. 6.B.



4. CONCLUSION

The objective of this study was to complete work of Tanku and Ceca (2013) in multidimensional density estimation and its application as an alternative method of empiric research. The purpose of research is to develop the tools and the metric that would allow researchers to interpret, analyze and compare empiric results of the multi-dimensional density estimates.

In this respect this paper defines cross section analysis and introduces it as the tool of interpretation and analysis. These cross sections represent conditional density of the "dependent variable" for all potential values of the explanatory variables. Numeric characteristics of the corresponding densities like mathematical expectation, standard deviation, variance etc, …, provide detailed information for the behavior of the dependent variable in response to changes in explanatory variables.

Further we apply this method to study the relationship between inflation and money in the case of US economy for a relatively long period of time which extends from 1959 to 2013. The purpose of empiric exercise was to test whether US monetary and inflation data support the monetarist view that Inflation is a monetary phenomenon. Our analysis show monetary and inflation developments during the last 6 decades do not provide empirical evidence to support this theoretic hypothesis.

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